J. Fluid Mech. (1980), *vol.* 100, *part* 1, *pp.* 35–40
Printed in Great Britain

A note on the drag on a slowly moving body in an axisymmetric rotating flow

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(Revised 21 May 1979 and in revised form 19 March 1980)

A variational expression is constructed for the drag on a sphere of radius a that moves with speed $U \ll \Omega a$ along the axis of a container of rotational speed Ω and length $h \gg a$ with $E = \nu/\Omega a^2 \ll 1$. Two complementary approximations, based on the asymptotic solutions in the limits $\delta = Eh/2a \uparrow \infty$ and $\delta \downarrow 0$, and a variational interpolation between these approximations are compared with the numerical results of Hocking, Moore & Walton (1979). The limit $\delta \downarrow 0$ is singular, and the variational principle fails in that limit in the sense that the error in the drag is of the same order as the error in the trial function rather than of second order therein; nevertheless, the variational interpolation is in error by less than 0·1 % for $\delta > 0{\cdot}003$ and by less than 1 % for all δ. The variational formulation may be of interest in other contexts.

1. Introduction

Hocking, Moore & Walton (1979; hereinafter referred to as HMW followed by the appropriate equation or section number therefrom) calculate the drag on a rigid sphere of radius a that moves with speed U along the axis of a rotating flow of angular speed Ω that is bounded by rigid, parallel planes a distance h apart on the assumptions that $U \ll \Omega a$, $E \equiv \nu/\Omega a^2 \ll 1$ (ν = kinematic viscosity), and $h \gg aE^{-\frac{1}{2}}$. These assumptions permit the boundary conditions on the sphere to be imposed on its equatorial projection, $0 \leqslant r < a, z = 0$ [so that the formulation presumably is valid for any axisymmetric body of length $l = O(a)$] and the perturbation flow to be expressed in terms of the discontinuity (across the equatorial plane) in azimuthal velocity, which I pose in the dimensionless form†

$$\zeta(s) \equiv \{v(r, 0-) - v(r, 0+)\}/(2U) \quad (s = r/a). \tag{1}$$

The ratio of the drag on the sphere to that for an unbounded flow is given by

$$\mathscr{D} \equiv D/(\tfrac{16}{3}\rho\Omega U a^3) = \frac{3\pi}{4}\int_0^1 \zeta(s)\, s^2\, ds. \tag{2}$$

† A development in terms of the pressure discontinuity

$$p(r, 0+) - p(r, 0-) = 4\rho U \Omega a \int_s^1 \zeta\, ds$$

leads to an equivalent but somewhat simpler formulation than that developed below; however, I have followed HMW in choosing $\zeta(s)$ as the basic dependent variable.

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The formulation of the resulting boundary-value problem is effected by HMW in terms of Fourier–Bessel integrals and culminates in a pair of dual integral equations for

$$Z(u) = \int_0^1 \zeta(s)\, J_1(us)\, s\, ds \equiv u^{-1}A(u), \tag{3}$$

the dimensionless Hankel transform of $\zeta(s)$. They then obtain numerical solutions by expanding Z in spherical Bessel functions and truncating the infinite set of equations for the expansion coefficients. They also obtain analytical approximations to $\mathscr{D}$ in the limits $\delta \downarrow 0$ and $\delta \uparrow \infty$, where $\delta = Eh/2a$. I present here a variational formulation, which is based on the integral equation for $\zeta(s)$ and provides direct approximations for $\mathscr{D}$.

2. Integral equation and variational principle

The substitution of (3) into the first of the dual integral equations for $A(u)$, HMW(2.15), which is derived from the boundary condition on the translational velocity of the sphere, yields the integral equation

$$\int_0^1 G(s,\sigma)\,\zeta(\sigma)\,\sigma\, d\sigma = \tfrac{1}{2}s \quad (0 \leqslant s \leqslant 1), \tag{4}$$

where

$$G(s,\sigma) = \int_0^\infty \Delta(u)\, J_1(us)\, J_1(u\sigma)\, du, \tag{5}$$

$$\Delta(u) = [1-\exp\{-(\delta_T+\delta_B)\,u^3\}]^{-1}\{1-\exp(-\delta_T u^3)\}\{1-\exp(-\delta_B u^3)\}, \tag{6a}$$

$$\delta_{T,B} \equiv \nu h_{T,B}/\Omega a^3 = 2\delta h_{T,B}/(h_T+h_B), \tag{6b}$$

and h_T (h_B) is the distance of the top (bottom) of the container from the equatorial plane. Note that $\Delta = \tanh(\frac{1}{2}\delta u^3)$ if $h_T = h_B$. The second of the dual integral equations, HMW(2.16), is equivalent to the boundary condition $\zeta(s) = 0$ in $s > 1$.

Multiplying (4) through by $s\zeta(s)$, integrating over $(0, 1)$, and dividing the result by the square of (2), I obtain

$$\frac{1}{\mathscr{D}} = \frac{\left(\dfrac{8}{3\pi}\right)\displaystyle\int_0^1\int_0^1 G(s,\sigma)\,\zeta(s)\,\zeta(\sigma)\, s\, ds\sigma\, d\sigma}{\left(\displaystyle\int_0^1 \zeta s^2\, ds\right)^2}, \tag{7}$$

which is a variational expression of Schwinger's type (Saxon & Schwinger 1968). It is stationary with respect to first-order variations of $\zeta(s)$ about the true solution to (4), exceeds the true value of $1/\mathscr{D}$ for any approximation to $\zeta(s)$ for which the above integrals exist (see appendix), and is invariant under a scale transformation of ζ. Alternative forms, based on the normalized trial function

$$\xi(s) = \left(\int_0^1 \zeta s^2\, ds\right)^{-1}\zeta(s), \tag{8}$$

are

$$\mathscr{D}^{-1} = \left(\frac{8}{3\pi}\right)\int_0^1\int_0^1 G(s,\sigma)\,\xi(s)\,\xi(\sigma)\, s\, ds\sigma\, d\sigma \tag{9a}$$

δ	$\mathscr{D}$ computed	(17), (18)	(12)	HMW(4.8)
10·0	1·00947	1·00947	1·00947	1·00946
6·0	1·0156	1·0156	1·0156	1·0156
4·0	1·0231	1·0231	1·0231	1·0231
2·5	1·0364	1·0364	1·0364	1·0362
1·5	1·0594	1·0594	1·0594	1·0589
1·0	1·0873	1·0873	1·0873	1·0862
0·6	1·1411	1·1411	1·1411	1·1383
0·4	1·2060	1·2060	1·2057	1·1995
0·25	1·3183	1·3183	1·3171	1·2987
0·15	1·5084	1·5084	1·503	
0·1	1·7345	1·7344	1·717	
0·06	2·1595	2·1592	2·09	
0·04	2·6538	2·6533	2·46	
0·025	3·4735	3·4725	2·96	
0·015	4·7977	4·7956	3·55	
0·010	6·3218	6·3180	4·09	

δ	$(128\delta/\pi)\mathscr{D}$ computed	(17), (18)	(14)	HMW(5.7)
0·006	2·2354	2·2336	1·31	1·85
0·004	2·0274	2·0256	1·26	1·74
0·0025	1·8368	1·8342	1·21	1·64
0·0015	1·6752	1·6713	1·17	1·54
0·001	1·5724	1·5674	1·15	1·47

TABLE 1. Comparison for $\delta_B = \delta_T = \delta$, of $\mathscr{D} \equiv D/D_0$ $(10 \geqslant \delta \geqslant 0{\cdot}01)$ and $(128\delta/\pi)\mathscr{D}$ $(0{\cdot}006 < \delta < 0{\cdot}001)$, as computed by Hocking, Moore & Walton (1979) using a truncation method, with HMW(4.8), HMW(5.7) and the present apprcximations

and

$$\mathscr{D}^{-1} = \left(\frac{8}{3\pi}\right)\int_0^\infty \Delta(u)\,\hat{Z}^2(u)\,du, \tag{9b}$$

where $\hat{Z}$ is the Hankel transform of ζ [cf. (3)].

The variational principle may be used to develop systematic approximations to ζ and $\mathscr{D}$ through the requirement that $1/\mathscr{D}$ be stationary with respect to variations of each of the coefficients in an appropriate expansion of $\zeta(s)$. In particular, the expansion of $s^{-1}(1-s^2)^{\frac{1}{2}}\zeta$ in Jacobi polynomials leads to the equivalent of HMW(3.6) and implies the monotonic increase, with the order of truncation, of $\mathscr{D}$ towards its exact value. Typically more valuable, however, is the construction of direct approximations to $\mathscr{D}$ through the positing of elementary or composite approximations to ζ.

3. Direct variational approximations

The solution of (4) in the limit $\delta \uparrow \infty$ yields

$$\zeta_1 = \tfrac{3}{2}s(1-s^2)^{-\frac{1}{2}}, \quad \hat{Z}_1 = \tfrac{3}{2}j_1(u), \tag{10a, b}$$

the substitution of which into (9*b*) yields the approximation

$$\mathscr{D}_1^{-1} = \frac{6}{\pi}\int_0^\infty \Delta(u) j_1^2(u)\, du, \tag{11}$$

where j_1 is a spherical Bessel function. The numerical evaluation of (11) is facilitated by separating out the limiting value $\Delta \sim 1$ (as $\delta \uparrow \infty$), as suggested by HMW. In particular, if $h_B = h_T$ ($\delta_B = \delta_T = \delta$), (11) may be transformed to

$$\mathscr{D}_1^{-1} = 1-(12/\pi)\int_0^\infty \{\exp(\delta u^3)+1\}^{-1} j_1^2(u)\, du. \tag{12}$$

The approximation (12) is in error by less than 0·1 % for $\delta \gtrsim 0{\cdot}1$ (see table 1); on the other hand (and not surprisingly), it fails as $\delta \downarrow 0$.

The solution of (4) in the limit $\delta \downarrow 0$ (see HMW § 5) yields

$$\zeta_2 = 12s(1-s^2), \quad \hat{Z}_2 = 24u^{-2}J_3(u), \tag{13a, b}$$

the substitution of which into (9*b*) yields the approximation

$$\mathscr{D}_2^{-1} = \frac{3{\cdot}2^9}{\pi}\int_0^\infty \Delta(u)\, J_3^2(u)\, u^{-4}\, du. \tag{14}$$

This approximation gives the correct limit as $\delta \downarrow 0$, but it is inaccurate in the neighbourhood of $\delta = 0$ (see table 1); in particular,

$$\mathscr{D}_2^{-1} = (128/\pi)\,\delta\{1-1{\cdot}256\delta^{\frac{1}{3}}+O(\delta^{\frac{2}{3}})\} \quad (\delta_B = \delta_T = \delta \downarrow 0). \tag{15}$$

This differs from HMW(5.7), which is of the same form with 1·256 replaced by 4·685.† It appears, then, that the variational principle fails in the limit $\delta \downarrow 0$ in the sense that the relative error in $\mathscr{D}$ is of the same order as the relative error in the trial function ζ, namely $\epsilon = O(\delta^{\frac{1}{3}})$, rather than $O(\epsilon^2)$ as the variational principle presumably implies [and as *is* true for (11)]; see § 5 and appendix for further discussion.

4. Composite variational approximation

The substitution of the trial function

$$\zeta = (\zeta_1+\alpha\zeta_2)/(1+\alpha) \tag{16}$$

into (9*a*) and the determination of α through the requirement $d\mathscr{D}^{-1}/d\alpha = 0$ yields the composite approximation

$$\mathscr{D} = \{\mathscr{D}_{11}+\mathscr{D}_{22}-2(\mathscr{D}_{11}\mathscr{D}_{22}/\mathscr{D}_{12})\}/\{1-(\mathscr{D}_{11}\mathscr{D}_{22}/\mathscr{D}_{12}^2)\}, \tag{17}$$

where $\mathscr{D}_{mn}$ is obtained by replacing $\zeta(s)\,\zeta(\sigma)$ by $\zeta_m(s)\,\zeta_n(\sigma)$ in (9*a*) or $\hat{Z}^2(u)$ by $\hat{Z}_m(u)\,\hat{Z}_n(u)$ in (9*b*). In particular, if ζ_1 and ζ_2 are given by (10*a*) and (13*a*),

$$\mathscr{D}_{11} = \mathscr{D}_1(11), \quad \mathscr{D}_{22} = \mathscr{D}_2(14), \quad \mathscr{D}_{12}^{-1} = \frac{96}{\pi}\int_0^\infty \Delta(u)\, j_1(u)\, J_3(u)\, u^{-2}\, du. \tag{18}$$

The approximation given by (17) and (18) with $h_B = h_T$ is in error by less than 0·1 % for $\delta > 0{\cdot}003$ (see table 1) and by less than 1 % for all δ. It exhibits the limiting form

† I get 4·723 using $\zeta(\frac{4}{3}) = -0{\cdot}9813$ (Jahnke & Emde 1943). I also get 4·72 through graphical extrapolation of HMW's computed values of $\mathscr{D}$.

(15), with 1·256 replaced by 4·092, as $\delta \downarrow 0$; accordingly, although quantitatively far superior to (14), it fails qualitatively in exactly the same manner.

5. The limit $\delta \downarrow 0$

The failure of the variational expression (7) to yield an error in $\mathscr{D}$ that is of second order in the error in the trial function in the limit $\delta \downarrow 0$ stems from the singular character of the kernel of the integral equation (4) in that limit (see appendix). It is reminiscent of a similar difficulty in the geometrical-optics limit for the variational formulation of the problem of diffraction through an aperture in a plane screen. A formulation based on the aperture field (Levine & Schwinger 1948) yields excellent results for moderate frequencies but is relatively unsuccessful for high frequencies, whereas a complementary formulation based on the discontinuity across the screen (Levine & Schwinger 1949) yields excellent results for high frequencies but fails qualitatively in the low-frequency limit. The latter formulation is rather more complicated, however, and it is not clear that it has an analog for the present problem. (I made a cursory attempt to obtain such a formulation, based on the axial velocity in the equatorial plane, and was confronted with divergent integrals.)

This work was partially supported by the Physical Oceanography Division, National Science Foundation, NSF Grant OCE77-24005, and by a contract with the Office of Naval Research.

Appendix. Proof of variational principle

Let $\zeta_*(s)$ be the true solution of the integral equation (4), $\hat{\zeta}_*$ be its normalized counterpart, as given by (8), and $\mathscr{D}_*$ be the corresponding value of $\mathscr{D}$, as given by (2). Now consider the positive-definite measure (of the error in $\hat{\zeta}-\hat{\zeta}_*$)

$$E = \left(\frac{8}{3\pi}\right)\int_0^1\int_0^1 G(s,\sigma)\{\hat{\zeta}(s)-\hat{\zeta}_*(s)\}\{\hat{\zeta}(\sigma)-\hat{\zeta}_*(\sigma)\}s\,ds\sigma\,d\sigma, \tag{A 1}$$

where $\hat{\zeta}(s)$ is a trial function, normalized as in (8), that is linearly independent of $\hat{\zeta}_*$ and for which the required integrals exist. The expansion of the integrand in (A 1) yields three distinct integrals, two of which are quadratic in $\hat{\zeta}$ and $\hat{\zeta}_*$, respectively, and are given by (9*a*) as

$$\int_0^1\int_0^1 G(s,\sigma)\,\hat{\zeta}(s)\,\hat{\zeta}(\sigma)s\,ds\sigma\,d\sigma = \left(\frac{3\pi}{8}\right)\mathscr{D}^{-1},$$
$$\int_0^1\int_0^1 G(s,\sigma)\,\hat{\zeta}_*(s)\hat{\zeta}_*(\sigma)s\,ds\sigma\,d\sigma = \left(\frac{3\pi}{8}\right)\mathscr{D}_*^{-1}; \tag{A 2a, b}$$

the third is given by

$$\int_0^1\int_0^1 G(s,\sigma)\,\hat{\zeta}_*(\sigma)\,\hat{\zeta}(s)s\,ds\sigma\,d\sigma = \frac{1}{2}\left\{\int_0^1 \zeta_*(s)\,s^2\,ds\right\}^{-1} = \left(\frac{3\pi}{8}\right)\mathscr{D}_*^{-1}, \tag{A 2c}$$

which follows from the invocation of (4) for ζ_*, (8) for $\hat{\zeta}$ and $\hat{\zeta}_*$, and (2) for $\mathscr{D}_*$. It then follows that

$$E = \mathscr{D}^{-1}-\mathscr{D}_*^{-1} > 0 \tag{A 3}$$

and hence that $\mathscr{D} < \mathscr{D}_*$, as stated in § 2.

It also follows from (A 1) and (A 3), after substituting G from (5), invoking (3) for $\hat{Z}$ and $\hat{Z}_*$, and dividing the result by $\mathscr{D}_*^{-1}$, as given by (9b), that

$$(\mathscr{D}_*/\mathscr{D})-1 = \int_0^\infty \Delta(\hat{Z}-\hat{Z}_*)^2\,du \Big/ \int_0^\infty \Delta\hat{Z}_*^2\,du. \tag{A 4}$$

This suggests that $(\mathscr{D}-\mathscr{D}_*)/\mathscr{D}_* = O(\epsilon^2)$ if $\zeta-\zeta_* = O(\epsilon)$, and this is true except for $\delta \downarrow 0$, in which limit G is singular, $\epsilon = \delta^{\frac{1}{3}}$, and (A 4) yields

$$(\mathscr{D}-\mathscr{D}_*)/\mathscr{D}_* = O(\epsilon^2\delta^{-\frac{1}{3}}) = O(\epsilon)$$

rather than $O(\epsilon^2)$.

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